

Valuation of Exotic Interest Rate Derivatives — Bermudans and Range Accruals

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Notation

- $Z(t, T)$ — Time t value of the zero coupon bond maturing at time T (for $t \leq T$).
- $f(t, T)$ — Forward rate for time T at time t .
- $r(t) = f(t, t)$ — Short rate (Forward rate for time t at time t)
- $B(t)$ — Time t value of the money market account.
- $R(t, T)$ — Rate of return on a zero coupon bond purchased at t and maturing at T .

Basic Relationships

$$R(t, T) = -\log Z(t, T)/(T - t)$$

$$f(t, T) = -\partial \log Z(t, T)/\partial T$$

$$r(t) = f(t, t)$$

$$= \lim_{T \rightarrow t} R(t, T)$$

$$B(t) = e^{\int_0^t r(s) ds}$$

$$Z(t, T) = e^{-\int_t^T f(t, s) ds}$$

In fixed income modeling, except for B , our rate and price processes have infinite variation and are not differentiable. On the other hand, $\log B$, being the integral of a process wrt time, is smooth. $\log Z$, in that it's integrating across *processes*, not time, is typically *not* smooth.

Basic Arbitrage Relationships

When Q is an equivalent Martingale measure with respect to the money market account B , we also have that:

$$\begin{aligned} Z(t, T) &= E_t^Q[Z(t', T)/B(t')]B(t) \\ &= E_t^Q[1/B(T)]B(t) \end{aligned}$$

so

$$Z(t, T)/B(t) = E_t^Q[e^{-\int_0^T r(s)ds}]$$

Choosing an arbitrage free system of zero coupon bonds is equivalent to choosing a short rate process and an equivalent measure, and systems of zero coupon bonds defined by choosing a short rate process and an equivalent measure are automatically arbitrage free. A system for which $Z(t, T)$ is increasing in T for all t is a system for which B is everywhere increasing, which is a system for which r is positive.

We can write down an SDE for the forward rates, zero coupon bonds, or for the spot rate, and convert between them:

If

$$dZ(t, T)/Z(t, T) = a(t, T)dt + \sigma(t, T)dW_t$$

then

$$\begin{aligned}df(t, T) &= (-a_2(t, T) + \sigma(t, T)\sigma_2(t, T))dt - \sigma_2(t, T)dW_t \\dr(t) &= (a_2(t, t) + \sigma(t, t)\sigma_2(t, t) + f_2(t, t))dt - \sigma_2(t, t)dW_t \\dB(t) &= r(t)B(t)dt\end{aligned}$$

(where a_2 is the derivative of a with respect to its second argument, etc).

Alternatively, if we start with the forward rates:

$$df(t, T) = a(t, T)dt + \sigma(t, T)dW_t$$

then

$$\begin{aligned}dZ/Z &= \left(r(t) - \int_t^T a(t, s)ds + \frac{1}{2} \left(\int_t^T \sigma(t, s)ds \right)^2 \right) dt \\ &\quad - \left(\int_t^T \sigma(t, s)ds \right) dW_t \\ dr(t) &= (a(t, t) + f_2(t, t))dt + \sigma(t, t)dW_t \\ dB(t) &= r(t)B(t)dt\end{aligned}$$

No arbitrage conditions

With $dZ/Z = a dt + \sigma dW$, under an EMM wrt numeraire N , with $dN(t)/N(t) = a_N(t)dt + \sigma_N(t)dW_t$, we have that:

$$a(t, T) - a_N(t) = (\sigma(t, T) - \sigma_N(t))\sigma_N(t).$$

Differentiating with respect to T yields:

$$a_2(t, T) = \sigma_2(t, T)\sigma_N(t).$$

When $\sigma_N(t) = 0$, this reduces to:

$$a(t, T) = a_N(t)$$

If the numeraire is the money market account, then a_N is the risk free rate, and this says that all zero coupon bonds under the EMM must have drift equal to the risk free rate.

In the latter case, then $a_2 = 0$, so the drift of the forward rates is then:

$$\sigma(t, T)\sigma_2(t, T)$$

No arbitrage conditions

With $df = a dt + \sigma dW$, under an EMM wrt numeraire N , with $dN(t)/N(t) = a_N(t)dt + \sigma_N(t)dW_t$, we have that:

$$\begin{aligned}r(t) &= \int_t^T a(t, s) ds + \frac{1}{2} \left(\int_t^T \sigma(t, s) ds \right)^2 \\ &= - \left(\int_t^T \sigma(t, s) ds + \sigma_N(t) \right) \sigma_N(t).\end{aligned}$$

Differentiating with respect to T yields:

$$-a(t, T) + \sigma(t, s) \int_t^T \sigma(t, s) ds = -(\sigma(t, T)) \sigma_N(t).$$

When $\sigma_N(t) = 0$, this reduces to the *HJM no arbitrage condition on the forward rates*:

$$a(t, T) = \sigma(t, s) \int_t^T \sigma(t, s) ds$$

Ring of equivalences

This gives a number of equivalent ways to specify an arbitrage free continuous interest rate model:

- Selecting a system of ZCBs $dZ(t, T)/Z(t, T) = a_Z(t, T)dt + \sigma_Z(t, T)dW_t$ with $a_Z(t, T) = r(t)$.
- Selecting a short rate process $dr(t) = a_r(t)dt + \sigma_r(t)dW_t$.
- Selecting a system of forward rates $df(t, T) = a_f(t, T)dt + \sigma_f(t, T)dW_t$, with $a_f(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, s)ds$.
- Selecting a numeraire N and a measure Q and defining $Z(t, T) = E_t^Q(1/N(T))N(t)$.

Valuation Philosophy

We take the following approach to valuation of derivatives.

- Fitting a model to prices and using it to value a security can be viewed as a combination of interpolation and extrapolation.
- Be explicit about the interpolation and extrapolation:
 - Value vanillas by interpolation.
 - Use models calibrated to the most appropriate vanillas to extrapolate the value of exotics.
- Simple models calibrated to closely related vanillas are better to use for valuing exotics than complex models.

The most appropriate vanillas to value an exotic are the ones that are most natural to use as hedges:

- The options that are most closely related to the exotic.
- The options that most closely replicate the volatility on which the exotic depends.

Pros and Cons of this Approach

Pros:

- Doesn't abstract market inputs, matches them.
- Can tune pricing of individual classes of exotics to match how they're traded.
- Yields simpler and more understandable hedges.
- Presumably faster to value simple options by simple means rather than all options under one sufficiently complex model.

Cons:

- Lack of uniform modeling makes relative valuation difficult.
- Substantial work required for each new structure.

While it's appealing to have a general model into which one can drop any derivative for valuation, it's also unrealistic. Without understanding the sources of volatility driving the derivative, it's likely that the valuation will not match the market. Especially considering that finite dimensional models will always have zero volatility invariants which the model will thus misprice.

Swaps

A swap is an agreement to exchange a fixed coupon for a floating coupon. We might pay fixed and receive float (a payer swap), or visa versa.

In a vanilla swap, the each legs cashflows are periodic according to date generation conventions. Coupons are computed based on day count and accrual conventions. Floating coupons are set on reset dates, typically two days before the start of the period.

- US standard — fixed coupons pay semiannually and the floating coupons pay quarterly.
- European standard — annual fixed coupons and semiannual floating coupons.

Swaps are typically entered into at the money, which is to say at zero cost, which is when the floating leg is worth the same amount as the fixed leg.

Swaps

In more detail, the data driving the swap specification looks like:

GRAB Equity **SWPM**

Options ▾ New Deal ▾ Save Deal ▾ View ▾ Hedge ▾ SWAP MANAGER

Deal Counterparty **SWAP_CNTRPARTY** Ticker / **SWAP** Series Deal # DETAIL

Receive Fixed ▾ DETAIL Pay Float ▾ DETAIL

Ticker	# SWAP	Series	Leg#	Ticker	# SWAP	Series	Leg#
Notional	10	MM	Cpn 4.12000 %	Notional	10	MM	Index US0003M
Curr	USD		Calc Basis Money Mkt	Curr	USD		Latest Index 5.14625
Effective	12/10/07		Pay Freq SemiAnnual	Effective	12/10/07		Spread 0.00 bp
Maturity	12/10/12		Day Cnt 30I 360	Maturity	12/10/12		Reset Freq Quarterly
FirstPmt	06/10/08		Unwind Cpn 4.12000 %	FirstPmt	03/10/08		Pay Freq Quarterly
NxtLastPmt	06/10/12		Unwind Annuity 0.00000 %	NxtLastPmt	09/10/12		Day Cnt ACT 360
DiscountCrv	23 Ask		USD Swaps(30/360,S/A)	DiscountCrv	23 Ask		USD Swaps(30/360,S/A)
				ForwardCrv	23 Ask		USD Swaps(30/360,S/A)

Valuation Curve **12/06/07** Valuation **12/10/07** All Values in USD

Market Value	10,000,000.00	DV01	4,486.33	Market Value	-10,000,000.01	DV01	-249.53
Accrued	0.00			Accrued	-0.00		

Net Principal	-0.01	Calculate Premium	Par Cpn	4.12000
Accrued	0.00	Premium	DV01	4,236.80
Market Value	-0.01	Unwind PV	0.00	

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Asia 61 2 9777 8600
Brazil 5511 3048 4500
Europe 44 20 7330 7500
Germany 49 69 920410
Hong Kong 852 2977 6000
Japan 81 3 3201 8900
Singapore 65 6212 1000
U.S. 1 212 318 2000

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Swap valuation

On a notional of 100 (par), if the floating coupon accrues at the discount rate, then on the fixing date, the floating leg is worth par. When the net value of the swap is zero, the coupon is the swap rate (the par rate for the given maturity and frequency). Thus, with payment dates t_i , and accrual rates α_i , for period i , the swap rate for maturity t_n , $S(t_n)$, satisfies:

$$100 = \sum_1^n 100S(t_n)\alpha_i Z(0, t_i) + 100Z(0, t_n)$$

so that

$$S(t_n) = (1 - Z(0, t_n)) / \sum \alpha_i Z(0, t_i)$$

Typically, the $S(t_n)$ are given for discrete times, and $Z(0, -)$ is computed from them via stripping assumptions.

Forward Libor rates

The forward Libor rate accruing from date d_1 to date d_2 , observed at date t is the simple interest rate equivalent of the forward discount factor $Z(t, d_2)/Z(t, d_1)$. As such, if the accrual period is α (approximately 1/4 for 3 month Libor, depending on exactly when the dates occur), then L satisfies:

$$1/(1 + L\alpha) = Z(t, d_2)/Z(t, d_1)$$

so that

$$L = \frac{1}{\alpha} \left(\frac{Z(t, d_1)}{Z(t, d_2)} - 1 \right)$$

Cap and Floor price quotation

A cap is a call option on the Libor rates over a sequence of dates, and hence is a portfolio of caplets (individual call options on specific libor rates).

Caps prices are quoted in implied volatility for discrete maturities and strikes.

GRAB										Corp TRVC																			
1<Go> to update																													
Interest Rate Volatility Cube																													
Currency USD					USD Bloomberg Cube					Caplet Smile Model					Piece-wise Linear														
Curve 3) 23					USD Swaps (30/360, S/A)					Caplet Stripping					Piece-wise Constant														
4) Export to Excel										Swaption Smile Model										PwL - Lift From Caplets									
Market Side	Bid	Contributors			BBIR	Date			11/26/07	8) Detail View																			
Term	Strike	ATM	1.00%	1.50%	2.00%	3.00%	3.50%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%															
1YR	4.14%	28.30	58.95	48.85	42.15	33.75	31.05	28.95	26.10	24.55	23.95	23.65	23.45	23.30															
2YR	3.75%	32.30	65.25	53.40	45.70	36.30	33.30	31.00	28.10	27.20	26.95	26.70	26.45	26.25															
3YR	3.88%	30.55	62.10	51.15	43.95	34.95	32.05	29.80	26.75	25.00	23.95	23.45	23.15	23.00															
4YR	4.04%	28.40	59.25	48.90	42.10	33.45	30.55	28.25	25.05	23.15	22.20	21.80	21.70	21.65															
5YR	4.19%	26.65	56.75	46.95	40.45	32.10	29.30	27.05	23.75	21.85	20.95	20.55	20.45	20.45															
6YR	4.31%	25.55	55.15	45.70	39.40	31.35	28.60	26.35	23.15	21.25	20.40	20.05	19.95	19.95															
7YR	4.41%	24.65	53.75	44.60	38.50	30.65	27.95	25.75	22.60	20.70	19.85	19.45	19.30	19.30															
8YR	4.51%	23.80	52.50	43.60	37.65	30.00	27.30	25.15	22.00	20.15	19.25	18.90	18.75	18.75															
9YR	4.57%	23.10	51.60	42.80	36.95	29.40	26.80	24.65	21.55	19.70	18.80	18.40	18.25	18.25															
10YR	4.64%	22.50	50.70	42.00	36.30	28.90	26.30	24.20	21.10	19.30	18.40	17.95	17.80	17.75															
12YR	4.74%	21.85	49.35	40.95	35.45	28.35	25.85	23.80	20.85	19.05	18.10	17.60	17.35	17.25															
15YR	4.83%	19.20	44.20	36.80	31.90	25.55	23.25	21.35	18.45	16.70	15.85	15.45	15.25	15.20															
20YR	4.92%	17.80	41.55	34.55	29.90	23.95	21.80	20.05	17.30	15.65	14.85	14.45	14.30	14.25															
25YR	4.95%	17.10	40.40	33.45	28.90	23.10	21.05	19.35	16.70	15.15	14.35	14.00	13.85	13.80															
30YR	4.95%	16.80	39.75	32.85	28.35	22.60	20.60	18.90	16.35	14.85	14.10	13.75	13.60	13.60															

Caplet valuation

Consider a caplet on 3 month Libor (L) maturing at time T , with a notional of 1 unit and strike K . Let α be the accrual period for this Libor rate (approximately 1/4). Then the caplet pays interest $L - K$ for the period if $L \geq K$ and thus has payoff at $T' = T + 3$ months of:

$$\alpha(L - K)^+$$

Under the equivalent Martingale measure Q with respect to a numeraire N , its value at time t is then:

$$E_t^Q[\alpha(L - K)^+ / N(T')]N(t)$$

If we choose N to be the value of a zero coupon bond maturing at time T' , then Q is the *forward measure* and the time t price of a caplet becomes:

$$E_t^{Q^F}[\alpha(L - K)^+]Z(t, T') = \alpha Z(t, T')E_t^{Q^F}[(L - K)^+]$$

Black Caplet Formula

Assuming the forward Libor rate at time t from T to T' ($L(t, T, T')$) under the forward measure follows geometric Brownian motion with constant instantaneous volatility σ , then the price is just the price of a call option under the forward based Black-Scholes Formula (AKA the Black Swaption Formula, AKA the Black Caplet Formula):

$$\text{BCF}(t, T, T', F, K, \sigma, \alpha) = Z(t, T')\alpha(FN(d_+) - KN(d_-)) \quad (1)$$

with

$$d_{\pm} = \frac{\ln(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2} \quad (2)$$

and $F = L(t, T, T')$.

While we can't expect to price all of the caplets at a fixed volatility, we can at least refer to their prices in terms of their implied volatilities, thus giving a volatility surface for the 3 month caplets as a function of strike and maturity.

Swaptions

A Swaption is an option to enter into a particular swap at a particular rate at a particular time in the future. The maturity of the swaption is called its *term*. The length of the underlying swap is its *tenor*. An option to enter into a 5 year swap in 1 year then has a term of 1 year and a tenor of 5 years and is called a 1 into 5 swaption.

Since (on a unit notional, paying fixed) a swap with rate K at time t with maturity T is roughly worth:

$$Z(t, T) - \left(\sum_1^n K \alpha_i Z(t, t_i) + Z(t, t_n) \right)$$

(Roughly because we're ignoring details like Libor fixing dates vs accrual period dates, floating legs not being par after fixing dates, etc).

Swaptions

Since the swaption is the option to enter into the swap at time T , the payoff of the swaption at expiration is:

$$\left(1 - \left(\sum_1^n K\alpha_i Z(T, t_i) + Z(T, t_n)\right)\right)^+$$

If S is the swap rate at this tenor at expiration, then

$$1 = \sum_1^n S\alpha_i Z(T, t_i) + Z(T, t_n)$$

so the swaption value in terms of the swap rate at time T is:

$$(S - K)^+ \sum_1^n \alpha_i Z(T, t_i)$$

So, except for the scaling factor, the swaption is essentially a call on the swap rate. Also, note that a one period swaption is a caplet.

Swaptions

Similar to caps, by assuming geometric Brownian motion with instantaneous volatility σ for the forward swap rate under the appropriate numeraire, we can get a simple formula for the value of the swaption.

In this case, the appropriate numeraire is

$$N(T) = \sum_1^n \alpha_i Z(T, t_i)$$

Then, the value of the swaption is

$$E_t^Q[(S - K)^+] \sum_1^n \alpha_i Z(t, t_i)$$

so if the forward swap rate at t for swaps from T to T' is $S(t, T, T')$, then a swaption with strike K has time t value of

$$(S(t, T, T')N(d_+) - KN(d_-)) \sum \alpha_i Z(t, t_i)$$

Volatility cube

We typically observe swap rates, cap and floors at specific maturities and at the money swaptions at particular maturities.

Stripping out the caplet vols from the caps, we're left with observing implied volatilities for discrete points in the term by tenor by strike space.

We deem options which are portfolios of caps and floors “Vanilla” options, and price them by interpolation — interpolate the implied vols of the underlying caps and floors from the volatility cube, and add up the prices of said options.

Cap stripping

Assume we have caps C_{ij} with

- strikes K_{ij} ,
- maturities t_i , and
- implied volatilities σ_{ij} .

If each cap is a “full” cap (i.e. - composed of a whole number of caplets), then we have times \bar{t}_i , and n_i such that $\bar{t}_{n_i} = t_i$ and the price of cap C_{ij} is

$$P_{ij} = \sum_{k=1}^{n_i} \text{BCF}(0, \bar{t}_k, \bar{t}'_k, L(0, \bar{t}_k, \bar{t}'_k), K_{ij}, \sigma_{ij}, \alpha_k)$$

Cap stripping

Of course, in a certain respect, this is internally inconsistent.

Typically, $K_{ij} = K_{i+1,j}$, and the only difference between C_{ij} and $C_{i+1,j}$ is the inclusion in the latter of the additional caplets maturing between t_i and t_{i+1} , yet the caplets they have in common are priced at different volatilities.

But this is a quoting convention. As long as it doesn't imply arbitrage (such as negative caplet prices) then it's fine.

The cap stripping process is finding a volatility surface $\bar{\sigma}(K, t)$ for the caplets which is consistent with the cap prices:

$$P_{ij} = \sum_{k=1}^{n_i} \text{BCF}(0, \bar{t}_k, \bar{t}'_k, L(0, \bar{t}_k, \bar{t}'_k), K_{ij}, \bar{\sigma}(K_{i,j}, \bar{t}_k), \alpha_k)$$

Cap stripping

If $n_{i+1} - n_i = 1$, and K_{ij} is independent of i (i.e. - if the cap volatilities are given in a discrete time/strike grid with times corresponding to the caplet maturities), then each cap consists of exactly one more caplet than the previous cap, and as long as their prices are increasing in maturity, then there exists a unique set of caplet prices.

This still doesn't completely solve the problem, because we need to know caplet prices for *all* times and strikes t , and K , not just the discrete ones, so we still need to do some sort of interpolation.

More generally, cap maturities increase in years, so each cap has four additional caplets, so the interpolation process itself is part of the stripping procedure.

Cap stripping

In general, we can assign a parameterized functional form to $\sigma(K, t)$, and try to find parameters which solve the above constraints, or ones which minimizes pricing or volatility errors.

Example:

Let our parameters be discrete caplet vols $\bar{\sigma}_{ij}$, which are caplet vols for strikes K_{ij} and maturity t_k .

Let $\bar{\sigma}(K, t)$ be defined by piecewise linear interpolation in K and piecewise constant interpolation in t .

This yields the same number of equations as unknowns and can be solved by bootstrapping if there's a solution.

Alternatively, the shape in strike can be improved a little by using cubic splines in K . Further improvement (at the loss of exactly matching all the prices) can be by fitting SABR in strike.

Alternative caplet volatility surfaces

One would think that the surface could be improved by piecewise linear interpolation in time, but this suffers from the same sort of sawtooth instabilities that plague yield curve stripping with piecewise linear forward rates.

Our experience is that the above method is far more robust and is still reasonable despite the unattractive discontinuous caplet volatilities. Another alternative is to do a global fit of a better functional form (such as monotone convexity splines). While this can lead to more handsome caplet vols, it's slow, complicated and has the interesting behavior that one cap vol moving can perturb the entire caplet surface.

Swaption smile lifting

To finish off the pricing of vanilla options, we need to incorporate pricing of swaptions in a way that's consistent with the observed swaption prices.

If we could observe the prices of swaptions of various maturities, tenors, and strikes, then we could interpolate their implied volatilities to price arbitrary swaptions.

However, often swaptions are just quoted at the money, so we need to estimate a swaption smile.

In that swaptions are essentially caplets of longer tenor, we do this by lifting the caplet smile to the swaptions.

Two lifting methods:

- Rescale the caplet vols until the caplet vol for the strike equal to the ATM swaption has the same volatility.
- Fit the SABR model to the caplets. Use SABR with the same parameters except replacing the ATM caplet rate with the ATM swaption rate.

Bermudan definition

A Bermudan swaption can be specified by a sequence of exercise dates T_i , a rate R , and a maturity T . The Bermudan swaption gives the holder the option at T_i to either enter into a swap at rate R maturing at time T , or continuing to hold the Bermudan swaption.

We denote by E_i , the option to enter into a swap at time T_i with rate R that matures at time T , and S_i the underlying swap. Then at T_n , the last exercise date, holding the Bermudan swaption is the same as holding E_n , and thus, between T_n and T_{n-1} , the price of E_n must be the same as the price of the Bermudan swaption.

At T_{n-1} , we have the option to either enter into the swap, or to not exercise and instead hold the swaption E_n . So, if $V_t(H)$ is the value of a claim or portfolio at time t , then at T_{n-1} the Bermudan swaption payoff is $B_{T_{n-1}} = \max(V_{T_{n-1}}(E_n), V_{T_{n-1}}(S_{T_{n-1}}))$. Similarly, at T_i , we have $B_{T_i} = \max(V_{T_i}(B_{T_{i+1}}), V_{T_i}(S_{T_i}))$.

Bermudan Swaption

For example:

GRAB Equity **SWPM**

Options | New Deal | Save Deal | View | SWAP MANAGER

Deal Counterparty **CS CNTRPARTY** Ticker **/ CS** Series Deal # DETAIL

Receive Fixed				Pay Float			
DETAIL				DETAIL			
Ticker	# CS	Series		Ticker	# CS	Series	
Notional	10 MM	Curr	USD	Notional	10 MM	Index	US0003M
Effective	12/10/07	Cpn	4.12000 %	Effective	12/10/07	Latest Index	5.14625
Maturity	12/10/12	Pay Freq	SemiAnnual	Maturity	12/10/12	Spread	0.00000 bp
Day Cnt	30I 360	<input type="checkbox"/>	Zero Coupon	Day Cnt	ACT 360	Reset Freq	Quarterly
DiscountCrv	23 Ask			ForwardCrv	23 Ask	Pay Freq	Quarterly

Option **Cancellable** **5 YR** **NC** **1 YR** DETAIL

Type **Bermudan** Position **Long** Model **LGM 1F**

First Call **12/08/08** Settlement **12/10/08** Vol Cube **VCUB** **USD Ask** **sabr**

Valuation Curve **12/06/07** Valuation **12/06/07** Settle **12/10/07**

Swap Premium	-0.00000	Calculate	Premium	Par Cpn	4.12000
Option Premium	2.26399	Delta (Hedge)	0.66340	Swap DV01	4,234.62
Switch Value	45,392.89	Gamma (10bps)	0.03332	Opt DV01	-1,711.49
Premium	2.26399	Vega (1%)	8,106.02	Yield Value	50.48
Market Value	226,398.74	Theta (1-day)	-128.20		

Refresh

Itals Cities Valuations Cariflow

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2007 Bloomberg Finance L.P.
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Bermudan Hedging

Whereas swaptions are vanilla instruments, only depending on the distribution of the swap rate at maturity, Bermudan swaptions are *not* vanilla, in that they're recursively options on options. So, we need a model to capture its value.

The Bermudan is a recursive combination of the values of the underlying European options E_j . Furthermore, the portfolio of the E_j dominates the payoff of the Bermudan.

Thus, using the E_j to hedge works well, and thus, a model for pricing the Bermudan should be calibrated to the E_j , and if it doesn't price the underlying E_j correctly, it's hard to imagine that the Bermudan will be properly priced.

Conversely, a model for the Bermudan that correctly prices the E_j only needs to capture the excess value of having the option to exercise in the future, which typically is much smaller in magnitude than the value of the options individually. So, it would be hard to make a tremendous overall pricing error by mispricing this additional component.

Bermudan Valuation

We value Bermudans using a one factor Hull-White model (AKA Linear Gaussian Markovian, AKA extended Vasicek model) calibrated to the underlying E_i .

The Hull-White model is typically presented (under the risk neutral measure with respect to the money market numeraire) in the following form:

$$dr = (\bar{\theta}_t - \kappa_t r_t)dt + \sigma_t dW_t$$

We can rewrite it as:

$$\begin{aligned}r_t &= \theta_t + X_t \\dX_t &= -\kappa_t X_t dt + \sigma_t dW_t\end{aligned}$$

(Just compute dr in the second set of equations and to get the first equation, one needs $\bar{\theta}_t = \theta'_t + \kappa_t \theta_t$, or $\theta = Ae^{\int \kappa_s ds} + \int \bar{\theta}_s e^{\int \kappa_u du} ds + \bar{\theta}_0$).

The LGM model is more commonly given by:

$$dX = \alpha_t dW_t$$

$$\zeta_t = \int_0^t \alpha_u^2 du$$

$$N(t) = \frac{1}{Z(0, t)} e^{H_t X_t + \frac{1}{2} H_t^2 \zeta_t}$$

$$Z(t, T) = E_t[1/N(T)]N(t)$$

The equivalence to the second form of HW is given by:

$$h_t = e^{-\int_0^t \kappa_u du}$$

$$H_t = \int_0^t h_u du$$

$$\alpha_t = \frac{\sigma_t}{h_t}$$

Bond prices under LGM

The convenience of the LGM formulation is evident in the calculation of the zero coupon bond process:

$$\begin{aligned}
 Z(t, T) &= E_t[1/N_T]N_t \\
 &= \frac{Z(0, t)}{Z(0, T)} E_t[e^{-H_T X_T - \frac{1}{2} H_T^2 \zeta_T}] e^{H_t X_t + \frac{1}{2} H_t^2 \zeta_t} \\
 &= \frac{Z(0, t)}{Z(0, T)} e^{H_t X_t + \frac{1}{2} (H_t^2 \zeta_t - H_T^2 \zeta_T)} E_t[e^{-H_T (X_T - X_t) - H_T X_t}] \\
 &= \frac{Z(0, t)}{Z(0, T)} e^{(H_t - H_T) X_t + \frac{1}{2} (H_t^2 \zeta_t - H_T^2 \zeta_T)} E_t[e^{-H_T (X_T - X_t)}] \\
 &= \frac{Z(0, t)}{Z(0, T)} e^{(H_t - H_T) X_t + \frac{1}{2} (H_t^2 \zeta_t - H_T^2 \zeta_T) - \frac{1}{2} H_T (\zeta_T - \zeta_t)}
 \end{aligned}$$

The latter equality comes from the fact that X_t is a Gaussian process with quadratic variation $\int_0^t \alpha^2(s) ds = \zeta(t)$, so $H_T(X_T - X_t)$ is normal has mean zero and variance $H_T(\zeta_T - \zeta_t)$.

We also note that this makes the zero coupon bond prices lognormally distributed.

Swap rates and Libor rates under LGM

Since the time t forward Libor rate for the period from T_1 to T_2 is

$$L = \frac{1}{\alpha} \left(\frac{Z(t, T_1)}{Z(t, T_2)} - 1 \right)$$

we have that

$$L = \frac{1}{\alpha} \left(\frac{Z(0, T_2)}{Z(0, T_1)} e^{sX_t + m} - 1 \right)$$

where

$$\begin{aligned} s &= (H_{T_2} - H_{T_1}) \\ m &= \frac{1}{2} (H_{T_2}^2 \zeta_{T_2} - H_{T_1}^2 \zeta_{T_1} + H_{T_2} (\zeta_{T_2} - \zeta_t) - H_{T_1} (\zeta_{T_1} - \zeta_t)) \end{aligned}$$

Thus, the Libor rates have a shifted lognormal distribution. Similarly, we also get an equation for the forward swap rate.

Caps under LGM

A caplet with strike K maturing at T_1 on a Libor rate L accruing from T_1 to T_2 has payoff

$$\alpha(L - K)^+$$

so its value is

$$E_t[\alpha(L - K)^+ / N(T_2)]N(t)$$

We could apply the formulas and work through the equations, but instead, we'll write down the formula for the swaption, since a caplet is just a one period swaption.

Swaptions under LGM

A swaption maturing at T on a receiver swap with coupon K and fixed leg cashflows t_i and accrual periods α_i has payoff

$$(K - S)^+ \sum_1^n \alpha_i Z(T, t_i)$$

The time T swap rate is given by

$$S = (1 - Z(T, t_n)) / \sum \alpha_i Z(T, t_i)$$

so the swaption payoff is:

$$\left(Z(T, t_n) - 1 + K \sum \alpha_i Z(T, t_i) \right)^+$$

Making use of Jamshidian's trick of decomposing an option on a portfolio into a portfolio of options (when each element of the portfolio is a monotonically decreasing function of the process), and skipping all the painful details, we get a formula that can't fit on this page.

Swaption formula under LGM

The price of a swaption under the LGM model is

$$Z(0, t_n)N\left(\frac{y^* + (H_{t_n} - H_T)\zeta_T}{\sqrt{\zeta_T}}\right) - N\left(\frac{y^*}{\sqrt{\zeta_T}}\right) \\ + \sum \alpha_i K Z(0, t_i)N\left(\frac{y^* + (H_{t_i} - H_T)\zeta_T}{\sqrt{\zeta_T}}\right)$$

where y^* satisfies:

$$1 = Z(0, t_n)e^{-(H_{t_n} - H_T)y^* - \frac{1}{2}(H_{t_n} - H_T)^2\zeta_T} \\ + \sum \alpha_i Z(0, t_i)e^{-(H_{t_i} - H_T)y^* - \frac{1}{2}(H_{t_i} - H_T)^2\zeta_T}$$

Calibration of the LGM model

To calibrate to the the E_i , we fix our mean reversion κ to be a constant. Then H is determined, and the only variable in the swaption price formula is ζ_T . So we can find ζ_{t_i} so that each E_i is priced properly by the model, and can then back out σ as a function of time by assuming it's piecewise constant.

Calibration of the LGM model

For example,

GRAB Equity **SWPM**

Options | New Deal | Save Deal | View | SWAP MANAGER

Deal Counterparty **CS CNTRPARTY** Ticker / CS Series Deal #

Model **LGM 1F**

Calibration Method **Diagonal Constant Kappa** Kappa **0.03**

Exercise Info Calibration Instruments

Expiry	Fixd Swap Value	European Value	Start	End	Strike	Volatility
12/08/08	-34147.23	134349.87	12/08/08	12/10/12	4.12000	26.39
06/08/09	-8704.46	155624.23	06/08/09	12/10/12	4.12000	26.02
12/08/09	47464.64	180700.93	12/08/09	12/10/12	4.12000	25.47
06/08/10	58047.72	174386.63	06/08/10	12/10/12	4.12000	25.21
12/08/10	66597.12	169744.14	12/08/10	12/10/12	4.12000	24.73
06/08/11	59025.00	130412.14	06/08/11	12/10/12	4.12000	24.34
12/08/11	46085.75	92255.64	12/08/11	12/10/12	4.12000	23.65

Valuation Curve **12/06/07** Valuation **12/06/07** Settle **12/10/07**

Swap Premium	-0.00000	Delta (Hedge)	0.66323	Par Cpn	4.12000
Option Premium	2.25858	Gamma (10bps)	0.03338	Swap DV01	4,234.62
Switch Value	45,156.73	Vega (1%)	8,106.89	Opt DV01	-1,711.05
Premium	2.25858	Theta (1-day)	-127.74	Yield Value	50.36
Market Value	225,857.65				<input type="button" value="Refresh"/>

Itals: Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410 Hong Kong 852 2977 6000
 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000
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Pricing of Bermudans

The final step to the pricing is the actual valuation of the Bermudan under the calibrated model.

If some lattice code happens to be laying around, it can be used by converting back to the standard HW form, but care must be taken to avoid introducing excess discretization error. This can be done by readjusting shifts to make sure the lattice is exactly calibrated to the yield curve. Vols can also be readjusted to match the calibrated model.

Alternatively, better accuracy can be gotten by making use of the fact that the model is Gaussian. In particular, $X_T - X_t$ is Gaussian with mean zero and variance $\zeta_T - \zeta_t$ and is independent of t . So, if V_T is a contingent claim on X_T , then $E_t[V_T]$ is the convolution of V_T with the normal density. We can compute it directly or via the FFT (based on the fact that the Fourier transform takes convolutions to products).

Range Accruals

Range accruals accrue a coupon R while a reference index (like 3 month Libor) is within a range.

Thus, the value of the range accrual coupon at a given date is R times the sum of the forward value of the digital options struck at the range points maturing on each day in the period.

If we can value the digitals, we can value the range accruals.

Range Accruals

For example:

GRAB Equity **SWPM**

Options | New Deal | Save Deal | View | SWAP MANAGER

Deal: **RACL_CNTRPARTY** | Ticker: **/RACL** | Series: | Deal #: | **DETAIL**

Receive **Callable Range Coupon** **DETAIL**

Notional	10 MM	Index	US0003M x 1.00000	Tenor	3 MO
Currency	USD	Latest Index	5.14625	Spread	0.00 bp
Effective	12/10/07	Reset Freq	Daily		
Maturity	12/10/12	Range Ceiling	6.00000 %	Range Floor	0.00000 %
Coupon	5.54382 %	Model	LGM 1F		
Pay Frequency	Quarterly	Discount Curve	23 Ask USD Swaps(30/360,S/A)		
Day Count	ACT 360	Forward Curve	23 Ask USD Swaps(30/360,S/A)		
Bus Day Adj	ModifiedFollowing	Vol Cube	VCUB USD Ask sabr		

Valuation Curve: 12/06/07 | Valuation: 12/10/07

Principal	9,782,015.65	Premium	97.82016	DV01	4,076.00
Accrued	0.00	Range Opt Prem	-2.14270	Vega(1%)	-11,270.71
Market Value	9,782,015.65	Call Opt Prem	-2.17984	Switch Value	-56,675.91
MV No Range	9,996,285.79	Workout Date	03/10/08		

Refresh

Itals: Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000

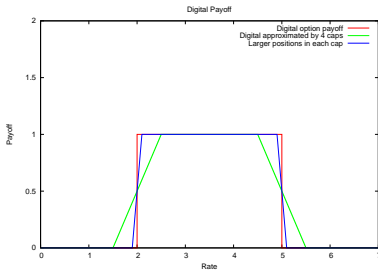
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Digital Options

The digital option pays one unit when a reference index (like 3 month Libor) is between two levels (L and H).

Consider buying a cap at $L - 1/2$, writing a cap at $L + 1/2$, writing a cap at $H - 1/2$, and buying a cap at $H + 1/2$. Then this payoff approximates the digital option payoff.

Consider buying 5 calls at $L - 1/10$, writing 5 calls at $L + 1/10$, writing 5 calls at $H - 1/10$, and buying 5 calls at $H + 1/10$. It's payoff is a better approximation of the digital.



In the limit, this becomes a digital, so we can price a digital based on the value of vanillas.

Digital formula

If $C(K)$ is the value of a caplet with strike K , then the value of the digital that pays 1 when $L \geq K$ is

$$\begin{aligned} & \lim_{N \rightarrow \infty} N(C(K - \frac{1}{2N}) - C(K + \frac{1}{2N})) \\ &= \lim_{h \rightarrow 0} \frac{(C(K - h) - C(K + h))}{2h} \\ &= -C'(K) \end{aligned}$$

The digital paying 1 between K and L is then

$$C'(L) - C'(K)$$

While this can be computed under the Black caplet model, that would ignore the fact that the implied volatility is changing with strike.

Rather than computing C' directly, it should be approximated by the difference derivative using the volatility cube. Care must be taken to use an appropriate h (depending on the accuracy of the computations in the volatility cube). If the computations are accurate to machine precision, then 10^{-5} is reasonable. Otherwise, a larger h is needed.

Back to Range Accruals

Since the range accrual coupon pays one unit (times R) at a later date T_i , we need the forward value at T_i of the sum of the digitals maturing between T_{i-1} and T_i . This needs two steps:

- Convexity correction of the daily range accruals.
- Interpolation and approximation to avoid computing 2520 digitals (and thus 10,080 vanillas) to compute the value of a range accrual maturing in 10 years.
- Summing the values of the coupons.

Valuing off-maturity caps

The value of the caplet on L from T to T' with strike K is (with respect to the forward measure for T')

$$\begin{aligned} E'[\alpha(L(T) - K)^+ / N'(T')] N'(0) \\ = E'[\alpha(L(T) - K)^+] N'(0) \end{aligned}$$

We know this value from the volatility cube, and hence can compute the value of digitals on $L(T)$ paying at T' .

However, the digitals underlying a range accrual all pay on the coupon date (T''), not on the ending accrual date. So, we need to adjust the prices.

The value of the same caplet, but paying on T'' instead of T' would be (under the forward measure for T'')

$$E''[\alpha(L(T) - K)^+] N''(0)$$

This is computed via a change of measure and an approximation (AKA a “convexity correction”).

Radon-Nikodym Derivative and Change of Measure

If P and Q are two absolutely continuous measures, then a random variable dP/dQ (the Radon-Nikodym derivative) exists and satisfies:

$$E^P[X] = E^Q[XdP/dQ]$$

for all random variables X .

If furthermore, P and Q are equivalent Martingale measures with respect to N and M , respectively, and the market in question is complete, then for all contingent claims X paying at time T , we have that

$$E^Q[Y/M(T)]M(0) = E^P[Y/N(T)]N(0)$$

so, taking $Y = XM(T)$, we have that:

$$E^Q[X] = E^P\left[X \frac{M(T)/M(0)}{N(T)/N(0)}\right]$$

so $\frac{M(T)/M(0)}{N(T)/N(0)}$ is the Radon-Nikodym derivative for \mathcal{F}_T measurable random variables, and gives us the multiplier necessary for changing measures.

Caplet Convexity Correction

Changing measure on the caplet paying on the coupon date, have

$$\begin{aligned} & E''[\alpha(L(T) - K)^+]N''(0) \\ &= E'[\alpha(L(T) - K)^+ \frac{N''(T)/N''(0)}{N'(T)/N'(0)}]N''(0) \end{aligned}$$

Since $N'(T) = Z(T, T')$ and $N''(T) = Z(T, T'')$, this yields

$$E'[\alpha(L(T) - K)^+ \frac{Z(T, T'')}{Z(T, T')}]Z(0, T')$$

In terms of Libor rates, if $L^*(T)$ is the forward Libor rate from T'' to T' , with accrual period β , then

$$\frac{Z(T, T'')}{Z(T, T')} = 1 + \beta L^*(T)$$

so the caplet price is

$$\begin{aligned} & E'[\alpha(L(T) - K)^+(1 + \beta L^*(T))]Z(0, T') \\ &= E'[\alpha(L(T) - K)^+]Z(0, T') + E'[\alpha(L(T) - K)^+\beta L^*(T)]Z(0, T') \end{aligned}$$

The first term on the right hand side is the unadjusted caplet price. The second term is the convexity correction that we have to estimate.

Estimating the Convexity Correction Term

To compute the convexity correction, we need to compute

$$\begin{aligned} & E'[\alpha(L(T) - K)^+ \beta L^*(T)] \\ &= E'[\alpha(L(T) - K)^+ \beta E[L^*(T)|L(T)]] \end{aligned}$$

The latter conditional expectation is a function of $L(T)$. We approximate it linearly, assuming it's of the form $a + bL(T)$.

To avoid changing the mean when doing this, so we want

$$\begin{aligned} E'(a + bL(T)) &= a + bL(0) \\ &= E'(L^*(T)) \\ &= L^*(0) \end{aligned}$$

So,

$$a = L^*(0) - bL(0)$$

(e.g. - If we're just approximating by shifting L , we should shift by the difference in the forward rates.)

Estimating the Convexity Correction Term

As for b , because $L^*(T)$ is a forward rate, it's determined by the relationship between the discount factors:

$$Z(T, T') = Z(T, T'')Z(T'', T')$$

Respecting this at time zero means that we need to forward $L(T)$ to T'' , so

$$b = Z(0, T)/Z(0, T'')$$

From this we can work out the formula for the convexity adjusted caplet, and then price the four caplets necessary for contributing to the cap.

This completes the valuation of the range accrual.

Callable Range Accruals

While range accruals are vanilla, callable range accruals are very much not vanilla.

To see how to price them, we need to think about their sources of volatility.

A swaption sees volatility from the variability of the forward swap rate.

A range accrual sees the same volatility, but it's relative to the percentage of time the observation index remains within bounds.

Thus, we need to capture the swap rate volatility, but at a strike corresponding to the percentage of value reduced by the range (i.e. - the digitals). We also need to capture the volatility of the digitals themselves, along with the correlation between the two.

We can do this in a two factor model by calibrating simultaneously to the implied volatility of the appropriately struck swaptions, and the value of the underlying digitals, and using the model correlation to generate the appropriate correlation between the observation index and the swap rate.

Callable Range Accruals

In a one factor model, we can't necessarily capture all of this simultaneously. In this case there are a number of approaches to their valuation via basic models:

- In the limit (as the range expands), they become bermudans, so calibrate to (and hedge with) the underlying Bermudans.
- Because they're options on the underlying digitals, calibrate to (and hedge with) the corresponding digitals.

Both approaches are not without their problems, leading to some adjustments coupled with the use of control variates applied to valuation (AKA adjusters) to correct for errors.

LGM1F RACL — Calibration

Calibrate to the underlying European swaptions. The only issue is to choose the strike. Options:

- Calibrate to the ATM swaptions.
- Calibrate to the swaptions at the effective coupon.
- Calibrate at the swaptions with the same moneyness — find the curve shift at which the option is at the money (the shift at which the value of the tail of the range accrual matches the present value of the call), and then calibrate to the ATM swaption on the shifted curve.

The first is easiest and will in the limit give the correct corresponding Bermudan price, but doesn't respect the smile.

The second makes sense in that the option to cancel the range accrual at a particular strike amounts to comparing the strike to the effective coupon at option maturity.

The third is an alternative that's similar to the second, but tries to take the relationship between the range and the rates into account. We've opted for the third approach, but are considering switching to the second. Both methods run into a certain amount of trouble as the range closes up.

Calibration results

Calibration of the range accrual curve leads to calibrating at a lower and variable coupon:

GRAB						EquitySWPM			
Options		New Deal		Save Deal		View		SWAP MANAGER	
Deal	Counterparty	RACL CNTRPARTY		Ticker	/ RACL		Series	Deal #	
Model		LGM 1F		Calibration Method		Diagonal Constant Kappa		Kappa	
Exercise Info		EXPORT TO EXCEL		Calibration Instruments		EXPORT TO EXCEL			
Expiry	Fwd Swap Value	European Value		Start	End	Strike	Volatility		
03/06/08	9763.41	-134611.90		03/06/08	12/10/12	4.03767	31.32		
06/06/08	32733.66	-167000.72		06/06/08	12/10/12	3.95630	29.13		
09/08/08	70286.67	-161247.84		09/08/08	12/10/12	3.87507	28.40		
12/08/08	111974.40	-151506.62		12/08/08	12/10/12	3.79857	27.48		
03/06/09	161660.21	-144439.43		03/06/09	12/10/12	3.72410	27.65		
06/08/09	193416.96	-133933.04		06/08/09	12/10/12	3.66362	27.76		
09/08/09	231060.94	-122307.08		09/08/09	12/10/12	3.60737	27.74		
Valuation		Curve		12/06/07		Valuation		12/10/07	
Principal	9,782,165.83		Premium	97.82166		DV01	4,075.25		
Accrued	0.00		Range Opt Prem	-2.14246		Vega (1%)	-11,259.30		
Market Value	9,782,165.83		Call Opt Prem	-2.17834		Switch Value	-56,586.33		
MV No Range	9,996,411.53						Refresh		
Italy		China		Valuation		Cashflow			
Australia 61 2 9777 8600		Brazil 5511 3048 4500		Europe 44 20 7330 7500		Germany 49 69 920410		Hong Kong 852 2977 6000	
Japan 81 3 3201 8900		Singapore 65 6212 1000		U.S. 1 212 318 2000		Copyright 2007 Bloomberg Finance L.P.		H283-220-0 06-Dec-07 12:27:36	

LGM1F RACL — Adjusters

Unfortunately, once the above is done, the calculation is off.

- When the option is worthless, the price doesn't become the price of the underlying range accrual.

The problem is that the model isn't calibrated to the right volatilities at the edges of the range, and hence the digital prices are off.

One way to "fix" this is via a control variate technique (AKA "Adjusters").

- Value each underlying range accrual coupon under the model.
- Compare to the known value.
- During the computation of the Bermudan price, rescale the cashflows of each coupon so that the time zero price under the model matches the known price.

This corrects the pricing of the underlying range accrual, but, in that it essentially assumes all of the error occurs at the coupon date, impacts the calls more than it should.

2 factor LGM

An alternative to the above would be to use a two factor LGM. In risk neutral short rate form with respect to the money market numeraire, this can be written as:

$$\begin{aligned}r_t &= \theta_t + X_t + Y_t \\dX_t &= -a_X X_t dt + \sigma_X(t) dW_1 \\dY_t &= -a_Y Y_t dt + \sigma_Y(t) dW_2 \\dW_1 dW_2 &= \rho dt\end{aligned}$$

2 factor LGM behavior

Typically, one will pick a high mean reversion for one factor, and a low mean reversion for the other.

- High mean reversion factor dampens long term effect of that parameter, leaving long tenor volatility mostly determined by low mean reversion factor.
- Both factors impact short tenor volatility.
- Negative correlation reduces volatility of short maturities relative to long maturities and allows this behavior to persist in time (time persistent volatility hump).

Rather than using a negative correlation to achieve a time persistent volatility hump, we'll use it to capture the right relationship between the swap rate and the observation index.

Calibration

Recall that there are three sources of volatility that impact the callable range accrual — the swap rate, the observation index, and the relationship between the two.

This inspires us to calibrate the two factor LGM as follows:

- Pick the mean reversions according to historical fitting.
- Pick the correlation to match the historical correlation between the swap rate and the observation index.
- Calibrate the volatilities to correctly price both the underlying Europeans struck at the effective coupon, and the digital coupons of the range accrual.

While we haven't tested this approach yet, similar work done for valuing callable spread range accruals indicates that this is a promising direction to pursue.

Spread Range Accruals

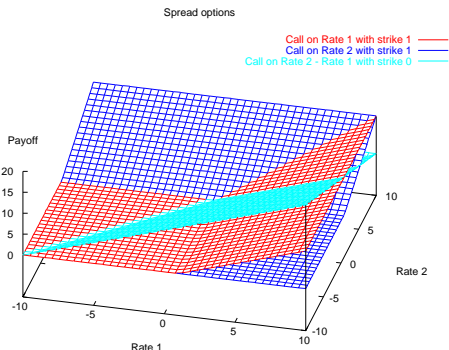
Spread RACLs are range accruals which accrue not on a pure rate, but ones which accrue on the difference between two spreads.

If we could observe prices of options on spreads, then these would be vanilla. Because these aren't traded or quoted, we need to apply modeling.

We can take a number of different approaches, but they all require calibrating one way or another to the value of options on spreads.

Options on spreads

Consider a call on $L_1 - L_2$, and compare to the payoff of a call on L_1 and a call on L_2 :



As such, the difference isn't in the span of the vanillas on the individual rates, so the difference cannot be well hedged by the vanillas, although this is worst when the two rates are uncorrelated, which is unusual. For example, the two year and ten year swap rates have a historical correlation of about 0.84.

Callable spread RACLs

Valuation using a short rate model is complicated by the difficulty in getting the sufficient volatility in the spread. The spread volatilities typically are extremely low in short rate models, thus grossly mispricing the underlying range accrual.

However, in the two factor LGM, we can sacrifice the time persistent hump, and choose a high correlation so that we can get sufficient spread volatility.

The calibration procedure then becomes:

- Fix the mean reversions by historical fittings (low and high), and the correlation at a high value (0.8).
- Pick the correlation to give the observed historical correlation between the swap rate and the observation index.
- Calibrate the volatilities to price the underlying swaptions at the effective rate as well as the digitals priced at the historically observed spread volatilities.

We've found this procedure to have reasonably good results.

Summary

- Valuation philosophy:
 - Value vanillas by interpolation.
 - Value exotics by “extrapolation” from related vanillas via a model.
 - Understand the sources of volatility that the exotic’s value is based on and model those sources well.
- Application of philosophy:
 - Caps, floors, and European swaptions via volatility interpolation (the volatility cube).
 - Digitals as portfolios of vanillas and hence also vanilla.
 - Bermudan swaptions via 1 factor LGM calibrated to diagonal swaptions.
 - Range accruals as vanillas, but needing convexity corrections.
 - Callable range accruals via 1 and 2 factor LGM with various calibrations and adjustments.
 - Spread range accruals and their valuation.

No arbitrage conditions

Suppose that there exists an equivalent Martingale measure Q with respect to numeraire N for the market of zero coupon bonds. Since $Z, N > 0$, if this is with respect to a Weiner space, we can write them as log processes:

$$\begin{aligned}dZ(t, T)/Z(t, T) &= a_Z(t, T)dt + \sigma_Z(t, T)dW \\dN/N &= a_Ndt + \sigma_NdW\end{aligned}$$

In this setting, Z/N must be a Martingale, so let's take a look at it's SDE.

$$\begin{aligned}d(Z/N)/(Z/N) &= (N/Z)(Zd(1/N) + (1/N)dZ + dZd(1/N)) \\ &= d(1/N)/(1/N) + dZ/Z + (dZ/Z)(d(1/N)/(1/N))\end{aligned}$$

To compute $d(Z/N)/(Z/N)$, we'll need to know $d(1/N)/(1/N)$:

$$\begin{aligned}d(1/N)/(1/N) &= Nd(1/N) \\ &= N(-N^{-2}dN + N^{-3}dNdN) \\ &= -dN/N + (dN/N)^2 \\ &= (\sigma_N^2 - a_N)dt - \sigma_NdW\end{aligned}$$

No arbitrage conditions

Thus,

$$\begin{aligned}d(Z/N)/(Z/N) &= (\sigma_N^2 - a_N)dt - \sigma_N dW \\ &\quad + a_Z dt + \sigma_Z dW \\ &\quad - \sigma_N \sigma_Z dt \\ &= (\sigma_N^2 - a_N - \sigma_N \sigma_Z + a_Z)dt + (\sigma_Z - \sigma_N)dW\end{aligned}$$

Assuming that $\int (Z/N)(\sigma_Z - \sigma_N)dW$ is a Martingale (as opposed to merely being a local Martingale), then for Z/N itself to be a Martingale, we must have that its drift term is zero, so, for all T , we must have that:

$$a_Z(t, T) - a_N(t) = (\sigma_Z(t, T) - \sigma_N(t)) \sigma_N(t).$$

No arbitrage, money market numeraire

If we take B as our numeraire,

$$\begin{aligned}dB &= e^{\int_0^t r(s)ds} d \int_0^t r(s)ds + \frac{1}{2} e^{\int_0^t r(s)ds} (d \int_0^t r(s)ds)^2 \\ &= Br(t)dt\end{aligned}$$

so, the numeraire volatility term is zero, and the no arbitrage condition reduces to:

$$a_Z(t, T) = a_N(t) = r(t).$$

This is exactly as one should expect because we're essentially in a Black-Scholes setting with Z being the stock and B the money market account, so under the equivalent Martingale measure, Z must have drift equal to the risk-free rate.

Forward measure case

If we choose a time T^* and use $Z(t, T^*)$ as our numeraire, we have that:

$$a_Z(t, T) - a_Z(t, T^*) = (\sigma_Z(t, T) - \sigma_Z(t, T^*))\sigma_Z(t, T^*).$$

Similarly to the money market case, we can choose the volatility of all our ZCBs, plus one drift, and the other drifts are determined.

No arb — forward rates

Working from the zero rates, we can work out the no arbitrage condition on the forward rates¹. $f = -\partial \log Z / \partial T$, so to find the SDE for f , we'll need the one for $\log Z$:

$$\begin{aligned}d \log Z(t, T) &= dZ/Z - dZdZ/(2Z^2) \\ &= dZ/Z - (1/2)(dZ/Z)^2 \\ &= (a_Z(t, T) - \sigma_Z^2(t, T)/2)dt + \sigma_Z(t, T)dW\end{aligned}$$

so

$$\begin{aligned}\log Z(t, T) &= \log Z(0, T) \\ &\quad + \int_0^t (a_Z(t, T) - \sigma_Z^2(t, T)/2)dt \\ &\quad + \int_0^t \sigma_Z(t, T)dW\end{aligned}$$

¹Due to Marcelo Piza

No arb — forward rates

Then our forward rates satisfy:

$$\begin{aligned}f(t, T) &= -\partial \log Z(t, T) \partial T \\ &= -\partial \log Z(0, T) \partial T \\ &\quad - \frac{\partial}{\partial T} \int_0^t (a_Z(t, T) - \sigma_Z^2(t, T)/2) dt \\ &\quad - \frac{\partial}{\partial T} \int_0^t \sigma_Z(t, T) dW\end{aligned}$$

No arb — forward rates

Assuming that a and σ are differentiable in T and sufficiently well behaved for the derivative to commute with the integrals, we then have that

$$\begin{aligned} f(t, T) &= -\partial \log Z(0, T) \partial T \\ &\quad - \int_0^t (a_{Z_2}(t, T) - \sigma_Z(t, T) \sigma_{Z_2}(t, T)) dt \\ &\quad - \int_0^t \sigma_{Z_2}(t, T) dW \end{aligned}$$

(where the subscript “2” refers to differentiation with respect to the second argument), so

$$\begin{aligned} df(t, T) &= (-a_{Z_2}(t, T) + \sigma_Z(t, T) \sigma_{Z_2}(t, T)) dt \\ &\quad - \sigma_{Z_2}(t, T) dW \end{aligned}$$

No arb — forward rates

Recall that the no arbitrage condition is that

$$a_Z(t, T) - a_N(t) = (\sigma_Z(t, T) - \sigma_N(t))\sigma_N(t).$$

Differentiating with respect to T yields:

$$a_{Z_2}(t, T) = \sigma_{Z_2}(t, T)\sigma_N(t).$$

so the SDE for f becomes:

$$\begin{aligned} df(t, T) &= (-\sigma_{Z_2}(t, T)\sigma_N(t) + \sigma_Z(t, T)\sigma_{Z_2}(t, T))dt \\ &\quad - \sigma_{Z_2}(t, T)dW \end{aligned}$$

If we define $\sigma_f(t, T) = -\sigma_{Z_2}(t, T)$, then

$$\begin{aligned} \int_t^T \sigma_f(t, s)ds &= \sigma_Z(t, t) - \sigma_Z(t, T) \\ &= -\sigma_Z(t, T) \end{aligned}$$

($Z(t, t) = 1$ implies $\lim_{s \rightarrow t} \sigma_Z(s, t) = 0$, so we can assume $\sigma_Z(t, t) = 0$).

No arb — forward rates

Rewriting df in terms of σ_f shows that when the model is arbitrage free, then f satisfies:

$$df(t, T) = \sigma_f(t, T) \left(\sigma_N(t) + \int_t^T \sigma_f(t, s) ds \right) dt + \sigma_f(t, T) dW$$

Thus, under the equivalent Martingale measure wrt to numeraire N , the forward rates must have drift a_f satisfying:

$$a_f(t, T) = \sigma_f(t, T) \left(\sigma_N(t) + \int_t^T \sigma_f(t, s) ds \right)$$

With respect to the money market numeraire (or any numeraire with zero volatility), we get the *HJM no arbitrage condition on the forward rates*:

$$a_f(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, s) ds$$

Short rate models

Before HJM, the most common way to specify an interest rate model was by defining the risk neutral dynamics of the short rate process r with respect to the money market numeraire.

Under the equivalent Martingale measure Q with respect to numeraire N_t , the zero coupon bond prices divided by the numeraire become Martingales:

$$\frac{Z(t, T)}{N_t} = E^Q \left[\frac{Z(t', T)}{N_{t'}} \mid F_t \right]$$

Setting $t' = T$ to make use of the fact that $Z(T, T) = 1$, and solving for Z , we get:

$$\begin{aligned} Z(t, T) &= E^Q [1/N_{t'} \mid F_t] N_t \\ &= E^Q [N_t/N_{t'} \mid F_t] \end{aligned}$$

In short rate models with numeraire being the money market account, this becomes:

$$\begin{aligned} Z(t, T) &= E^Q \left[e^{-\int_0^T r(t)dt} \mid \mathcal{F}_t \right] e^{\int_0^t r(t)dt} \\ &= E^Q \left[e^{-\int_t^T r(t)dt} \mid \mathcal{F}_t \right] \end{aligned}$$

Hull-White model rate process

The SDE for the short rate under the Hull-White model can be solved. Using the second form of the SDE, let $g(t) = \int_0^t a_s ds$. Multiplying the SDE for X by $e^{g(t)}$ and noting that

$$d(e^g X) = e^g dX + e^g a_t X_t dt$$

we can integrate and substitute to find that:

$$r(t) = \theta_t + (r_0 - \theta(0))e^{-g(t)} + \int_0^t e^{g(s)-g(t)} \sigma_s dW_s,$$

so r is a Gaussian process, with mean $\theta_t + (r_0 - \theta(0))e^{-g(t)}$ and variance $\int_0^t e^{2(g(s)-g(t))} ds$. Then $\int_0^T r(t) dt$ will also have a Gaussian distribution, whose mean and variance can be computed, so $Z(t, T)$ will be a lognormal process for each T .